Boundedness and blow-up of solutions for a nonlinear elliptic system arising in probability and stochastic processes

Dragos-Patru Covei

Constantin Brancusi University of Tg-Jiu, Gorj, Romania

E-mail: covei.dragos@yahoo.com

Abstract

The main objective in the present paper is to obtain the existence results for bounded and unbounded solutions of some quasilinear elliptic systems. Related results as obtained here have been established recently in [C. O. Alves and A. R.F. de Holanda, *Existence of blow-up solutions for a class of elliptic systems*, Differential Integral Equations, Volume 26, Number 1/2 (2013), Pages 105-118.]. Also, we present some references to give the connection between these type of problems with probability and stochastic processes, hoping that these are interesting for the audience of analysts likely to read this paper.

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1 Introduction

The question of existence of solutions for elliptic equation of the form

$$\Delta_p u = f(x, u) \text{ in } \Omega, \tag{1.1}$$

was studied by many researchers (see Bandle [2], the author [4], Lair [11], Matero [14], Mohammed [16] with their references). This work is devoted to the study of the more general nonlinear elliptic problems of the type

$$\begin{cases} \Delta_p u_i = F_{u_i}(x, u_1, ..., u_i, ..., u_d) \text{ in } \Omega, \\ i = 1, ..., d \end{cases} \tag{1.2}$$

where $d \geq 1$ is integer, $\Omega \subset \mathbb{R}^N (N > 1)$ is a smooth, bounded domain or $\Omega = \mathbb{R}^N$, $\Delta_p u_i := \text{div}\left(|\nabla u_i|^{p-2} \nabla u_i\right) (1 < p < \infty)$ is the p-Laplacian operator and $F_{u_i} (i = 1, ..., d)$ stands for the derivatives of a continuously differentiable function $F : \Omega \times [\mathbb{R}^+]^d \to \mathbb{R}^+$ in $(u_1, ..., u_d)$. For the case $\Omega = \mathbb{R}^N$, we also consider the following class of elliptic systems:

$$\begin{cases} \Delta_p u_i = a_i(x) F_{u_i}(x, u_1, ..., u_i, ..., u_d) \text{ in } \mathbb{R}^N, \\ u_i > 0 \text{ in } \mathbb{R}^N, \\ i = 1, ..., d \end{cases} \tag{1.3}$$

where $a_i : \mathbb{R}^N \to (0, \infty)$ are suitable functions. Associated with the class of systems (1.3), our main result is concerned with the existence of entire large solutions, that is, solutions $(u_1, ..., u_d)$ satisfying $u_i(x) \to \infty$ as $|x| \to \infty$ for all $i = 1, ..., d$.

The interest on systems (1.2)-(1.3) comes from some problems studied in the works of Lasry-Lions [12], Busca-Sirakov [3] and Dynkin [9] where the authors give the connection between these type of problems with probability and stochastic processes and from the recently work of Alves and Holanda [1] where these systems are considered for the case $d = p = 2$, in terms of the pure mathematics. The difference between our work and the paper by [1] is that: our systems can have any number of equations, the potential functions a_i cover more general properties and that we use in the proofs theories for quasilinear operators instead of the theories for linear operators used by Alves-Holanda [1].

To begin with our results we make the following convention: we say that a function $h : [0, +\infty) \to [0, \infty)$ belongs to $\mathcal{F}$ if

$$\begin{aligned} h & \in C^1([0, \infty)), \ h(0) = 0, \ h'(t) \geq 0 \ \forall t \in [0, \infty), \\ h(t) & > 0 \ \forall t \in (0, \infty) \end{aligned}$$

and the Keller-Osserman [10], [17] condition is satisfied, that is,

$$\int_1^\infty \frac{1}{H(t)^{1/p}} dt < \infty,$$

where $H(t) = \int_0^t h(s) \, ds$.

Our main result for problem (1.2) on a bounded domain is the following:

Theorem 1.1. *Suppose Ω is a smooth, bounded domain in $\mathbb{R}^N$ and that there exist $f_i, g \in \mathcal{F}$ satisfying*

$$F_{t_i}(x, t_1, ..., t_i, ..., t_d) \geq f_i(t_i) \ \forall x \in \overline{\Omega}, \ t_i > 0 \ and \ i = 1, ..., d \tag{1.4}$$

and

$$g(t) \geq \max\{F_t(x, t, ..., t)\} \ \forall x \in \overline{\Omega}, \ t > 0. \tag{1.5}$$

Then:

1. problem (1.2) admits a positive solution with boundary condition

$$\begin{cases} u_i = \alpha_i \ on \ \partial\Omega \\ i = 1, ..., d \\ \alpha_i \in (0, \infty). \end{cases} \tag{1.6}$$

2. problem (1.2) admits a positive solution with the boundary condition

$$\begin{cases} u_i = \infty \ on \ \partial\Omega \\ i = 1, ..., d \end{cases} \tag{1.7}$$

where $u_i = \infty$ on $\partial\Omega$ should be understood as $u_i(x) = \infty$ as $dis(x, \partial\Omega) \to 0$.

3. problem (1.2) admits a positive solution with boundary condition: there are $i_0, j_0 \in \{1, ..., d\}$ such that

$$\begin{cases} u_{i_0} = \infty \ on \ \partial\Omega \\ u_{j_0} < \infty \ on \ \partial\Omega \ for \ any \ j_0 \neq i_0 \end{cases} \tag{1.8}$$

and the set $\{1, ..., d\}$ is crossed by i_0 respectively j_0.

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Our next result is related to the existence of a solution for system (1.3). For expressing the next result, we assume that functions a_i $(i = 1, ..., d)$ satisfy the following conditions:

$$a_i(x) > 0 \text{ for all } x \in \mathbb{R}^N \text{ and } a_i \in C_{loc}^{0,\vartheta}(\mathbb{R}^N), \vartheta \in (0, 1) \tag{1.9}$$

and that the quasilinear system

$$-\Delta_p z(x) = \sum_{i=1}^{d} a_i(x) \text{ for } x \in \mathbb{R}^N, \ \ z(x) \to 0 \text{ as } |x| \to \infty \tag{1.10}$$

has a C^1-upper solution, in the sense that

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla z|^{p-2} \nabla z \cdot \nabla \phi dx \geq \int_{\mathbb{R}^N} \sum_{i=1}^{d} a_i(x) \phi dx, \ \phi \in C_0^\infty(\mathbb{R}^N), \ \phi \geq 0 \\ z \in C^1(\mathbb{R}^N), \ z > 0 \text{ in } \mathbb{R}^N, \ z(x) \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{1.11}$$

Theorem 1.2. *Assume that (1.4)-(1.11) hold. Then system (1.3) has an entire large C^1-solution (in the distribution sense).*

To prepare for proving our theorems, we need some additional results.

2 Preliminary results

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a smooth, bounded domain in $\mathbb{R}^N$ and $1 < p < \infty$. The first auxiliary result can be seen in the paper of Matero [15, pp. 233].

Lemma 2.1. *Assume that g meets the conditions: g is a continuous, positive, increasing function on $\mathbb{R}_+$, and $g(0) = 0$. Let $h \in W^{1,p}(\Omega)$ be such that $(G \circ h) \in L^1(\Omega)$ where $G(s) = \int_0^s g(t) dt$. Then there exists a unique $u \in W^{1,p}(\Omega)$ which (weakly) solves the problem*

$$\begin{cases} \Delta_p u(x) = g(u(x)), \ x \in \Omega, \\ u(x) = h(x), \ x \in \partial\Omega. \end{cases}$$

Furthermore, if u_1 and u_2 are the solutions corresponding to h_1 and h_2 with $h_1 \leq h_2$ on $\partial\Omega$, then $u_1 \leq u_2$ in Ω. Finally, there exists an $\alpha \in (0, 1)$ such that $u \in C^{1,\beta}(D)$ $(\beta \in (0, 1))$ for any compact set $D \subset \Omega$.

The following comparison principle is proved in the article of Sakaguchi [18] (or consult some ideas of the proof in work of Tolksdorf [19, Lemma 3.1.]).

Lemma 2.2. *Let $u, v \in W^{1,p}(\Omega)$ satisfy $-\Delta_p u \leq -\Delta_p v$ for $x \in \Omega$, in the weak sense. If $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.*

The following Lemma is essentially and can be found in Zuodong ([22]).

Lemma 2.3. *Suppose $f \in \mathcal{F}$ and that $u \in W_{loc}^{1,p}(\Omega) \cap C(\Omega)$ satisfies*

$$-\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \int_\Omega f(u) \varphi dx, \forall \varphi \in C_0^\infty(\Omega).$$

Then, there exists a monotone decreasing function $\mu : (0, \infty) \to (0, \infty)$ determined by f such that

$$u(x) \leq \mu(dist(x, \partial\Omega)) \ \forall x \in \Omega.$$

Moreover,

$$\lim_{t \to 0} g(t) = \infty, \lim_{t \to \infty} g(t) = -\infty.$$

Next, we begin with recalling the definition of sub and super-solution used in the present context. The system that we will study is the following

$$
\begin{cases}
\Delta_p u_i = G_{u_i}\left(x, u_1, ..., u_i, ..., u_d\right) \text{ in } \Omega \\
u_i = f_i \text{ on } \partial\Omega \\
i = 1, ..., d
\end{cases}
\tag{2.1}
$$

where $f_i \in W^{1,p}(\Omega)$ and $G\left(x, t_1, ..., t_i, ..., t_d\right) : \Omega \times [\mathbb{R}]^d \to \mathbb{R}$ is measurable in $x \in \Omega$, continuously differentiable in $t_i \in \mathbb{R}$, and satisfies the following condition: for each $T_i > 0$ fixed $(i = 1, ..., d)$, there exists $C = C\left(T_i\right) > 0$ such that

$$
\left|G\left(x, t_1, ..., t_d\right)\right| \leq C \ \forall \left(x, t_1, ..., t_d\right) \in \Omega \times [-T_i, T_i]^d.
\tag{2.2}
$$

Now we introduce the concept of sub- and super-solution in the weak sense.

Definition 2.1. *By definition* $(\underline{u}_1, ..., \underline{u}_d) \in [W^{1,p}(\Omega)]^d$ *is a (weak) sub-solution to (2.1), if* $\underline{u}_i \leq f_i$ *on* $\partial\Omega$,

$$
\int_\Omega |\nabla \underline{u}_i|^{p-2} \nabla \underline{u}_i \nabla \phi dx - \int_\Omega G_{\underline{u}_i}\left(x, \underline{u}_1, ..., \underline{u}_d\right) \phi dx \leq 0
$$

for all $\phi \in C_0^\infty(\Omega)$ *with* $\phi \geq 0$ *and* $i = 1, ..., d$.

Similarly $(\overline{u}_1, ..., \overline{u}_d) \in [W^{1,p}(\Omega)]^d$ *is a (weak) super-solution to (2.1) if in the above the reverse inequalities hold.*

The following result holds:

Lemma 2.4. *Suppose* $(\underline{u}_1, ..., \underline{u}_d)$ *is a sub-solution while* $(\overline{u}_1, ..., \overline{u}_d)$ *is a super-solution to problem (2.1), and assume that there are constants* $\underline{a}_i, \overline{a}_i \in \mathbb{R}$ *such that*

$$
\underline{a}_i \leq \underline{u}_i \leq \overline{u}_i \leq \overline{a}_i \text{ almost every where in } \Omega.
$$

If (2.2) holds, then there exists a weak solution $(u_1, ..., u_d) \in [W^{1,p}(\Omega)]^d$ *of (2.1), satisfying the condition*

$$
\underline{u}_i \leq u_i \leq \overline{u}_i \text{ almost everywhere in } \Omega.
$$

We will not give the proof here since he can now proved as in [1, Theorem 2.1, pp. 110] with some ideas from [21].

3   Proof of main results

In this section, we will prove the main results of this paper.

3.1   Proof of Theorem 1.1

3.1.1   Proof of 1:

In what follows, we denote by $\psi \in W^{1,p}(\Omega)$ the unique positive solution of the problem

$$
\begin{cases}
-\int_\Omega |\nabla\psi|^{p-2} \nabla\psi \nabla\phi dx = \int_\Omega g\left(\psi\right) \phi dx \text{ in } \Omega, \ \forall \phi \in C_0^\infty(\Omega) \text{ with } \phi \geq 0, \\
\psi > 0 \text{ in } \Omega \\
\psi = m \text{ on } \partial\Omega
\end{cases}
$$

where $m = \min \{\alpha_1, ..., \alpha_d\}$, which exists and minimizes the Euler-Lagrange functional

$$J(\psi) = \int_\Omega \left(\frac{1}{p} |\nabla \psi|^p + G(\psi(x)) \right) dx$$

on the set

$$K = \left\{ v \in L^1(\Omega) \, \middle| \, v - m \in W_0^{1,p}(\Omega) \text{ and } (G \circ v) \in L^1(\Omega) \right\}$$

i.e., ψ meets the boundary condition $(\psi - m) \in W_0^{1,p}(\Omega)$ in the weak sense [see Lemma 2.1 and [2, Paragraph 6, pp. 12]]. Then,

$$\begin{cases} -\int_\Omega |\nabla \psi|^{p-2} \nabla \psi \nabla \phi dx = \int_\Omega g(\psi) \phi dx \geq \int_\Omega F_\psi(x, \psi, ..., \psi) \phi dx \text{ in } \Omega, \\ \psi = m \leq \alpha_i \text{ on } \partial\Omega \\ i = 1, ..., d \end{cases}$$

and so $(\underline{u}_1, ..., \underline{u}_d) = (\psi, ..., \psi)$ is a sub-solution. Clearly, $(\overline{u}_1, ..., \overline{u}_d) = (M, ..., M)$, with

$$M = \max \{\alpha_i \, | i = 1, ..., d\},$$

is a super-solution of (1.2) with the boundary condition (1.6). We prove that, $\underline{u}_i \leq \overline{u}^i$ for all $i = 1, ..., d$. Indeed,

$$\begin{cases} -\Delta_p \underline{u}_i = -g(\underline{u}_i) \leq -\Delta_p \overline{u}_i = 0 \text{ in } \Omega, \\ \underline{u}_i = m \leq \overline{u}_i = M \text{ on } \partial\Omega, \\ i = 1, ..., d \end{cases}$$

and then with the use of Lemma 2.2 it follows that $\underline{u}_i \leq \overline{u}_i$ in $\overline{\Omega}$. According with the proof of the Lemma 2.4, there exists a critical point $(u_1, ..., u_d) \in [W^{1,p}(\Omega)]^d$ which minimize the Euler-Lagrange functional

$$I(u_1, ..., u_d) = \frac{1}{p} \int_\Omega \sum_{i=1}^d |\nabla u_i|^p \, dx + \int_\Omega F(x, u_1, ..., u_d) \, dx$$

that solve, in the weak sense, the system

$$\begin{cases} \Delta_p u_i = F_{u_i}(x, u_1, ..., u_i, ..., u_d) \text{ in } \Omega \\ u_i = \alpha_i \text{ on } \partial\Omega \\ i = 1, ..., d \end{cases} \tag{P_α}$$

and satisfying $\psi \leq u_i \leq M$ in Ω for all $i = 1, ..., d$. Since $u_i \in L^\infty_{loc}(\Omega)$, by the regularity theory [5, 13, 20], it follows that $u_i \in C^1(\Omega)$.

3.1.2 Proof of 2:

To study this case, we begin considering the system

$$\begin{cases} \Delta_p u_i = F_{u_i}(x, u_1, ..., u_i, ..., u_d) \text{ in } \Omega, \\ u_i = n \text{ on } \partial\Omega, \\ i = 1, ..., d. \end{cases} \tag{3.1}$$

Then, by the finite case above, problem (3.1) has a solution $(u_1^n, ..., u_d^n)$.

We prove that the sequence of solutions $(u_1^n, ..., u_d^n)$ can be chosen satisfying the inequality

$$u_i^n \leq u_i^{n+1} \text{ for all } i = 1, ..., d \text{ and } n \in \mathbb{N}. \tag{3.2}$$

To prove this, we consider the solution $(u_1^1, ..., u_d^1)$ of the problem

$$\begin{cases} \Delta_p u_i = F_{u_i}(x, u_1, ..., u_i, ..., u_d) \text{ in } \Omega, \\ u_i = 1 \text{ on } \partial\Omega, \\ i = 1, ..., d. \end{cases} \tag{3.3}$$

and note that it is a sub-solution of

$$\begin{cases} \Delta_p u_i = F_{u_i}(x, u_1, ..., u_i, ..., u_d) \text{ in } \Omega, \\ u_i = 2 \text{ on } \partial\Omega, \\ i = 1, ..., d. \end{cases} \tag{3.4}$$

while the pair $(M_1, ..., M_1)$ is a super-solution for $M_1 = 2$. Once $0 \leq u_i(x) \leq 2$ $(i = 1, ..., d)$ $\forall x \in \overline{\Omega}$, Lemma 2.4 implies that there exists a solution $(u_1^2, ..., u_d^2)$ of

$$\begin{cases} \Delta_p u_i = F_{u_i}(x, u_1, ..., u_i, ..., u_d) \text{ in } \Omega, \\ u_i = 2 \text{ on } \partial\Omega, \\ i = 1, ..., d. \end{cases}$$

satisfying $u_i^1(x) \leq u_i^2(x)$. Using the argument above, for each $M_n = n + 1$; $n = 1, 2, ...$, we get a solution $(u_1^n, ..., u_d^n)$ of (3.1), which is a sub-solution, and the pair $(M_n, ..., M_n)$ is a super-solution respectively of

$$\begin{cases} \Delta_p u_i = F_{u_i}(x, u_1, ..., u_i, ..., u_d) \text{ in } \Omega, \\ u_i = n + 1 \text{ on } \partial\Omega,, \\ i = 1, ..., d. \end{cases}$$

Thereby, the sequence of solutions $(u_1^n, ..., u_d^n)$ satisfies the inequality (3.2). Finally, we construct an upper bound of the sequence. More exactly, we show that $\{(u_1^n, ..., u_d^n)\}_{n \geq 1}$ is uniformly bounded in any compact subset of Ω. To this end, we begin recalling that by (1.4)

$$\begin{cases} \Delta_p u_i^n \geq f_i(u_i^n) \text{ in } \Omega \\ u_i^n > 0 \text{ in } \Omega \\ u_i^n \leq n \text{ on } \partial\Omega \end{cases}$$

with $f_i \in \mathcal{F}$. If $\widetilde{u}_i^n$ $(i = 1, ..., d)$ denote the unique solutions of the problems

$$\begin{cases} \Delta_p u_i = f_i(u_i) \text{ in } \Omega \\ u_i > 0 \text{ in } \Omega \\ u_i \leq n \text{ on } \partial\Omega \end{cases}$$

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it follows from Lemma 2.2 that

$$u_i^n \leq \widetilde{u}_i^n \text{ in } \Omega \text{ for all } n \geq 1.$$

By Lemma 2.3, there exist non-increasing continuous functions $\mu_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\widetilde{u}_i^n \leq \mu_i \left(dist \left(x, \partial\Omega \right) \right) \ \forall n \in \mathbb{N}, \forall x \in \Omega \text{ and } i = 1, ..., d$$

showing that

$$0 < u_i^1 \left(x \right) \leq u_i^n \left(x \right) \leq \mu_i \left(d \left(x \right) \right) \ \forall n \in \mathbb{N}, \forall x \in \Omega \tag{3.5}$$

where $d \left(x \right) = dist \left(x, \partial\Omega \right)$. Thus there exists a subsequence, still denoted again by u_i^n, which converges to a function u_i in $W^{1,p} \left(\Omega \right)$. In other words

$$u_i \left(x \right) := \lim_{n \to \infty} u_i^n \left(x \right) \text{ for all } x \in \Omega \text{ and } i = 1, ..., d.$$

The estimates (3.5) combined with the bootstrap argument yield that $u_i^n \left(x \right) \to u_i \left(x \right)$ in $C^1 \left(K \right)$ for any compact subset $K \subset \Omega$. Furthermore, it is clear that, $u_i \left(x \right) \in C^1 \left(\Omega \right)$ and $\left(u_1, ..., u_d \right)$ is a solution of (1.2); that is,

$$\begin{cases} \Delta_p u_i = F_{u_i} \left(x, u_1, ..., u_i, ..., u_d \right) \text{ in } \Omega, \\ \\ u_i > 0 \text{ in } \Omega, \\ \\ i = 1, ..., d. \end{cases}$$

To complete the proof, it suffices to prove that $\left(u_1, ..., u_d \right)$ blows up at the boundary. Supposing for the sake of contradiction that u_i does not blow up at the boundary, there exist $x_0 \in \partial\Omega$ and $\left(x_k \right) \subset \Omega$ such that

$$\lim_{k \to \infty} x_k = x_0 \text{ and } \lim_{k \to \infty} u_i \left(x_k \right) = L_i \in \left(0, \infty \right).$$

In what follows, fix $n > 4L_i$ and $\delta > 0$ such that $u_i^n \left(x \right) \geq n/2$ for all $x \in \overline{\Omega}_\delta$, where

$$\overline{\Omega}_\delta = \left\{ x \in \overline{\Omega} \, | dist \left(x, \partial\Omega \right) \leq \delta \right\}.$$

Then, for k large enough, $x_k \in \overline{\Omega}_\delta$ and $u_i^n \left(x_k \right) > 2L_i$. Since

$$u_i^n \left(x_k \right) \leq u_i^{n+1} \left(x_k \right) \leq ... \leq u_i^{n+j} \left(x_k \right) \leq ... \leq u_i \left(x_k \right) \forall j,$$

we have that $u_i \left(x_k \right) \geq 2L_i$, which is a contradiction. Therefore, u_i blows up at the boundary. This solution $\left(u_1, ..., u_d \right)$ dominates all other solutions and is therefore commonly called blow-up/large solution.

3.1.3 Proof of 3:

Let $\left(..., u_{i_0}^n, ..., u_{j_0}^n, ... \right) \in C^1 \left(\Omega \right)$ be the solution of the problem $\left(P_\alpha \right)$ with $\alpha_{i_0} = n$, $n \in N$, and α_{j_0} fixed. As in the previous case, the sequence $u_{i_0}^n$ is bounded on a compact subset contained in Ω, implying that there exist functions u_{i_0} $\left(i = 1, ..., d \right)$ satisfying $u_{i_0}^n \to u_{i_0}$ in $C^1(K)$ $\left(i = 1, ..., d \right)$ for any compact subset $K \subset \Omega$. Moreover, the arguments used in the previous cases yield that u_{i_0} blows up at the boundary, that is, $u_{i_0}^n = \infty$ on $\partial\Omega$. Related to the sequence $\left(u_{j_0} \right)$, we recall that

$$\begin{cases} \Delta_p u_{j_0}^n = F_{u_{j_0}} \left(x, u_1^n, ..., u_{j_0}^n, ..., u_d^n \right) \text{ in } \Omega, \\ \\ u_{j_0}^n = \alpha_{j_0} \text{ on } \partial\Omega. \end{cases}$$

Then, by the comparison principle $u_{j_0}^n \leq \alpha_{j_0} \ \forall x \in \overline{\Omega}$ and $n \geq 1$. Passing to the limit as $n \to \infty$, we obtain that $u_{j_0} \leq \alpha_{j_0}$ for all $x \in \Omega$.

Claim. Let $x_0 \in \partial\Omega$ and $\left(x_k \right) \subset \Omega$ be a sequence with $x_k \to x_0$. Then $u_{j_0} \left(x_k \right) \to \alpha_{j_0}$ as $k \to \infty$.

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Indeed, if the limit does not hold, there exist $\varepsilon > 0$ and a subsequence of (x_k), still denoted by itself, such that

$$x_k \to x_0 \text{ and } u_{j_0}(x_k) \le \alpha_{j_0} - \varepsilon \ \forall k \in \mathbb{N}. \tag{3.6}$$

Since $u_{j_0} = \alpha_{j_0}$ on $\partial\Omega$ and is continuous, there is some $\delta > 0$ such that $u_{j_0}(x_k) \ge \alpha_{j_0} - \frac{\varepsilon}{2}$, $\forall x \in \overline{\Omega}_\delta$. Hence, for k large enough, $x_k \in \overline{\Omega}_\delta$ and $u_{j_0}^1 \ge u_{j_0}(x_k) \ge \alpha_{j_0} - \frac{\varepsilon}{2} > \alpha_{j_0} - \varepsilon$ which contradicts (3.6). From this claim, we can continuously extend the function u_{j_0} from Ω to $\overline{\Omega}$ by considering $u_{j_0}(x) = \alpha_{j_0}$ on $\partial\Omega$, concluding this way the proof of the Finite and infinite case.

3.2 Proof of Theorem 1.2

Firstly, we provide a sub-solution for the problem (1.3). To do this we consider the function $w : \mathbb{R}^N \to [0, \infty)$ implicitly defined by

$$z(x) = \int_{w(x)}^{\infty} \frac{1}{g^{1/(p-1)}(t)} dt, \ x \in \mathbb{R}^N.$$

Note that $w \in C^1\left(\mathbb{R}^N, (0, \infty)\right)$, $w(x) \to +\infty$ as $|x| \to \infty$ and

$$\nabla z(x) = -g^{-1/(p-1)}(w(x)) \nabla w(x) \tag{3.7}$$
$$|\nabla w(x)|^{p-2} \nabla w(x) = -g(w(x)) |\nabla z(x)|^{p-2} \nabla z(x). \tag{3.8}$$

Given $\phi \in C_0^\infty\left(\mathbb{R}^N\right)$, $\phi \ge 0$ we have

$$\int_{\mathbb{R}^N} |\nabla w(x)|^{p-2} \nabla w(x) \nabla \phi dx = \int_{\mathbb{R}^N} -g(w(x)) |\nabla z(x)|^{p-2} \nabla z(x) \nabla \phi dx$$
$$= \int_{\mathbb{R}^N} \text{div} \left[g(w(x)) |\nabla z(x)|^{p-2} \nabla z(x)\right] \phi dx.$$

Computing the derivatives in the integrand of the expression just above, in the distribution sense, using (3.7) we get,

$$\int_{\mathbb{R}^N} |\nabla w(x)|^{p-2} \nabla w(x) \nabla \phi dx = \int_{\mathbb{R}^N} g(w(x)) \Delta_p z(x) \phi dx$$
$$- \int_{\mathbb{R}^N} g'(w(x)) g^{\frac{1}{p-1}}(w(x)) |\nabla z(x)|^p \phi dx.$$

Using the fact that $g \in \mathcal{F}$ and that $z(x)$ is an upper solution of (1.11) we derive the inequality

$$\int_{\mathbb{R}^N} |\nabla w(x)|^{p-2} \nabla w(x) \nabla \phi dx - \int_{\mathbb{R}^N} g(w(x)) \Delta_p z(x) \phi dx \le 0,$$

and so

$$\int_{\mathbb{R}^N} |\nabla w(x)|^{p-2} \nabla w(x) \nabla \phi dx + \int_{\mathbb{R}^N} g(w(x)) \left(\sum_{i=1}^{d} a_i(x)\right) \phi dx \le 0,$$

which together with (1.5) leads to

$$- \int_{\mathbb{R}^N} |\nabla w(x)|^{p-2} \nabla w(x) \nabla \phi dx \ge \int_{\mathbb{R}^N} a_i(x) F_{u_i}(x, w(x), ..., w(x)) \phi dx$$

for all $i = 1, ..., d$.

In the next, we consider the system

$$\begin{cases} \Delta_p u_i = a_i(x) F_{u_i}(x, u_1, ..., u_i, ..., u_d) \text{ in } B_n \\ u_i = w_n \text{ in } \partial B_n, \\ i = 1, ..., d, \end{cases} \tag{3.9}$$

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where B_n is the open ball of radius n centered at the origin and $w_n = \max_{x \in \overline{B}_n} w(x)$. Clearly, $(w, ..., w)$ and $(w_n, ..., w_n)$ are a sub-solution and super-solution for (3.9) respectively. Thus, by Theorem 1.1, there is a solution $(u_1^n, ..., u_d^n) \in [W^{1,p}(B_n)]^d$ of (3.9) satisfying $w(x) \le u_i^n \le w_n$ for all $x \in \overline{B}_n$ and $i = 1, ..., d$. By the comparison Lemma $u_i^n \le u_i^{n+1}$ and moreover $u_i^n \in C^1(B_n)$. For $m \ge 1$ and $n \ge m+1$ consider the family of systems

$$\begin{cases} \Delta_p u_i^n = a_i^a f_i(u_i^n) & \text{in } B_{m+1} \\ \\ i = 1, ..., d, \end{cases}$$

where $a_i^a = \min_{x \in \overline{B}_n} a_i(x) > 0$. Arguing as in the previous sections, there are monotone decreasing functions $\mu_i^a : (0, \infty) \to (0, \infty)$ determined by f_i such that

$$w(x) \le u_i^n(x) \le \mu_i^a(dist(x, \partial B_{m+1})) \ \forall x \in B_{m+1}$$

from which it follows that

$$w(x) \le u_i^n(x) \le M_i^m \text{ for all } n \in \mathbb{N}, x \in \overline{B}_m, i = 1, ..., d$$

and for some positive constants M_i^m. As a consequence, there is $u_i \in L^\infty(B_m)$ $(i = 1, ..., d)$ such that for some sub-sequence of u_i^n, still denoted by itself, we get

$$u_i^n \to u_i \ (i = 1, ..., d) \text{ pointwisely in } B_m \ (\forall m > 1).$$

Therefore, $(u_1, ..., u_d) \in C^1(\mathbb{R}^N)$ and is a solution for the system

$$\begin{cases} \Delta_p u_i = a_i(x) F_{u_i}(x, u_1, ..., u_i, ..., u_d) & \text{in } \mathbb{R}^N \\ \\ u_i > 0 \text{ in } \mathbb{R}^N \\ \\ i = 1, ..., d \end{cases}$$

satisfying

$$w(x) \le u_i(x) \text{ for all } x \in \mathbb{R}^N \text{ and } i = 1, ..., d. \tag{3.10}$$

Letting $|x| \to \infty$ in (3.10) it follows that $(u_1, ..., u_d)$ is a large entire solution for (1.3).

4 Remarks

4.1 The probabilistic connection

Assume that ψ belongs to a wide class Ψ of monotone increasing convex functions. There is an area in probability theory where boundary-blow-up problem

$$\begin{cases} \Delta u = \psi(u) & \text{in } \Omega \\ \\ u = \infty & \text{on } \partial \Omega \end{cases}$$

arise (see the paper [6] or directly the book [7] for details). The area is known as the theory of superdiffusions, a theory which provides a mathematical model of a random evolution of a cloud of particles. Indeed, given any bounded open set Ω in the N-dimensional Euclidean space, and any finite measure μ we may associate with these the exit measure from Ω i.e. (X_Ω, P_μ), a random measure which can be constructed by a passage to the limit from a particles system. Particles perform independently Δ-diffusions and they produce, at their death time, a random offspring (cf. [8]). P_μ is a probability measure determined by the initial mass distribution μ of the offspring and X_Ω corresponds to the instantaneous mass distribution of the random evolution cloud. Then procedding in this way, one can obtain any function ψ from a subclass Ψ_0 of Ψ which contains u^γ with $1 < \gamma \le 2$. Dynkin [6], also provided a simple probabilistic representation of the solution for the class of problems u^γ $(1 < \gamma \le 2)$, in terms of the so-called exit measure of the associated superprocess. Moreover, the author say that a probabilistic interpretation is known only for $1 < \gamma \le 2$.

4.2 The stochastic connection

In the next we present a simple example from the paper Lasry-Lions [12] in order to connect our problems with stochastic processes. To do this, we consider the stochastic differential equation

$$\begin{cases} dX_t = a_t dt + dB_t \\ X_0 = x \in \Omega \end{cases}$$

where B_t is the Brownian motion (on a standard probability space) and $a_t = a(X_t)$ is a feedback control. We also consider the class $\mathcal{A}$ of all (feedback) controls that keep the diffusion process X_t inside the domain Ω for any time $t > 0$ a.s.. The criterion for optimality is given by the cost functional (E is the expected value, $C_q = (q-1)q^{-q'} > 0$ and $q' = \frac{q}{q-1}$):

$$J(x,a) = E \int_0^\infty \left\{ \underbrace{f(X_t)}_{\text{assigned cost}} + \underbrace{C_q |a(X_t)|^{q'}}_{\text{cost of the control}} \right\} \underbrace{e^{-t} dt}_{\text{discount factor}}.$$

Then

$$\inf_{a \in \mathcal{A}} J(x,a)$$

where

$$A = \left\{ a \in C^0\left(\Omega, R^N\right) : X_t \in \Omega, \forall t > 0 \text{ a.s.} \right\}$$

is achieved and defines the value function

$$u(x) = \inf_{a \in \mathcal{A}} J(x,a)$$

that solves, in the case $1 < q \le 2$ (dynamic programming principle), the problem

$$\begin{cases} \Delta u + f(x) = u + |\nabla u|^q \\ u(x) \to \infty \text{ as } d(x) \to 0 \end{cases} \tag{4.1}$$

where $d(x) = dist(x, \partial\Omega)$.

We also remark from the work of Busca-Sirakov [3] that the above disscussion for the scalar equation (4.1) suggest that the solutions of the system (1.1.) can be viewed as the value function of a stochastic control process, and the boundary conditions then means that the process is discouraged to leave the domain by setting an infinite cost on the boundary. For a more detailed discussion about practical applications where such problems appear we advise the reader the introduction of the work [14].

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